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AD 607019

THE DISTRIBUTION OF RADIAL ERROR AND
ITS STATISTICAL APPLICATION IN WAR GAMING

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SUMMARY

This research memorandum presents a unified treatment of the assumptions and theorems concerning the distribution of radial error, and demonstrates its statistical application in war gaming. The density function and cumulative distribution function of the radial error are derived and graphed for one, two, and three dimensions. For each of these cases, formulas are given for the expectation, standard deviation, and median of the radial error. Tables of pertinent conversion are provided. Results are given for the distribution of detonation points in three dimensions that are useful in war games employing atomic rockets or torpedoes.

LIST OF SYMBOLS

Symbol		Definition	Equation Ref.
a_n	=	ratio of α_n to σ	(36)
c_n	=	normalizing constant in $f_n(r)$	(3)
$D(\cdot)$	=	standard deviation of $\cdot$	
d_n	=	ratio of σ_n to σ	(20)
$E(\cdot)$	=	expectation of $\cdot$	
e_n	=	ratio of μ_n to σ	(15)
$F_n(r)$	=	cumulative distribution of radial error R_n	(4)
$f_n(r)$	=	density function of radial error R_n	(2)
$g(x)$	=	standard Gaussian density function	(1)
$g^{(k)}(x)$	=	k-th derivative of $g(x)$	
$H_n(s)$	=	cumulative distribution function of standardized radial error S_n	(31)
$h_n(s)$	=	density function of standardized radial error S_n	(32)
$I(t,u)$	=	incomplete Gamma function ratio	
$J(x)$	=	integral of $g(t)$ from 0 to x	
$k_n(x)$	=	chi-square density function with n degrees of freedom	
$M(\cdot)$	=	median of $\cdot$	
n	=	dimension of rectangular coordinate space	
$P(\cdot)$	=	probability of $\cdot$	
$Q_k(t)$	=	Hermite polynomial of order k	
R_n	=	radial error random variable	
r	=	dummy variable	
S_n	=	standardized radial error random variable	
s	=	dummy variable	

TABLE OF CONTENTS

LIST OF SYMBOLS (Continued)

Symbol		Definition	Equation Ref.
t	=	dummy variable	
u	=	dummy variable	
$V(\cdot)$	=	variance of $\cdot$	
v_n	=	ratio of σ_n^2 to σ^2	(18)
X_i	=	i-th random coordinate	
x_i	=	i-th rectangular coordinate	
y_i	=	product of σ and x_i	(38)
α_n	=	median of R_n	(37)
$\Gamma(t)$	=	Gamma function	
$\Gamma_t(u)$	=	incomplete Gamma function	
μ	=	expectation of X	
μ_n	=	expectation of R_n	(14)
σ^2	=	variance of X	
σ_n^2	=	variance of R_n	(17)
σ_i'	=	standard deviation of i-th coordinate in ellipsoidal case	

I. INTRODUCTION

When studying a physical system, it is frequently necessary to devise a mathematical model involving the distribution of the length of a random vector whose components are independently, normally distributed about the origin with variance σ^2. A model of this type is often useful in the physical sciences and has been employed by operations analysts in the theory of bombing error. This research memorandum defines such a model, investigates its theoretical properties, and applies it to the statistical distribution of weapon radial error in war gaming.

In war gaming when the desired ground zero for an atomic weapon is given, the actual ground zero can be statistically determined by random sampling. The statistical technique for doing this is elementary, but no one has bothered to spell it out in a step-by-step manner for the use of persons unfamiliar with statistics. Moreover, the three-dimensional case has not been fully treated previously despite the fact that atomic rockets and torpedoes are critical weapons in many detailed war games. Consequently, it is believed that a unified treatment of the assumptions and theorems concerning the distribution of weapon radial error is needed for ready reference. Formulas, derivations, graphs, and constants for the one, two, and three-dimensional cases are collected here to aid the operations analyst, and an example is given to illustrate the application of the statistical model.

It is assumed that the n rectangular coordinates of the detonation point are mutually independent and that each has a Gaussian distribution with expectation zero and variance σ^2. Thus the model is based on the n-dimensional sperical Gaussian distribution since the coordinates have a

common variance. In an ellipsoidal model where the standard deviations σ_i' are unequal we have the approximation

$$\sigma \doteq (\sigma_1' \ . \ . \ . \ \sigma_n')^{1/n} .$$

This is a good approximation when the standard deviations are nearly equal, and it is still adequate when the largest standard deviation is twice the smallest. For extreme ellipsoids, however, the above approximation should not be used.

The statistical method of determining the detonation point requires just three inputs: the aiming point, the direction of approach to the target,* and the means of delivery of the weapon. From the first two we construct a rectangular coordinate system at the aiming point with one axis oriented along the approach direction. From a knowledge of the means of weapon delivery we choose a median radial error. The median radial error in the one, two, and three-dimensional cases is, respectively, the linear probable error, the circular probable error, and the spherical probable error. Using a conversion factor derived here, the median radial error is converted into a standard deviation. Finally, each of the n coordinates is selected from a table of random Gaussian numbers corresponding to this standard deviation.

Generally the terminology and notation used here conform to that of current books on probability and statistics. For example, following the modern trend, random variables are capitalized to distinguish them from ordinary variables.

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*Mathematically, this input is required only in the ellipsoidal model; however, because of tactical considerations it is often useful in the spherical model.

II. DISTRIBUTION OF THE RADIAL ERROR

To find the distribution of the radial error in one, two, and three dimensions we will derive the general function and then set n equal to 1, 2, and 3. Let $\mathcal{G}(X;\ \mu, \sigma^2)$ denote that the random variable X has a Gaussian distribution with expectation μ and variance σ^2, i.e., the density function for X is

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$$

Let

$$g(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \qquad -\infty < x < \infty \qquad (1)$$

be the density function of a standard Gaussian random variable X. Also let $\chi^2(X;\ n)$ denote that the random variable X has a Chi-square distribution with n degrees of freedom, i.e., the density function for X is

$$k_n(x) = \frac{1}{\Gamma(n/2)2^{n/2}}\, x^{n/2-1}\, e^{-x/2} \qquad x \geq 0.$$

Assuming

$$\mathcal{G}(X_i;\ 0,\ \sigma^2) \qquad \text{for } i = 1, \ldots, n$$

we see by division that

$$\mathcal{G}(X_i/\sigma;\ 0,\ 1) \qquad \text{for } i = 1, \ldots, n.$$

But the square of a standard Gaussian random variable has a Chi-square distribution with one degree of freedom, i.e.,

$$\chi^2(X_i^2/\sigma^2;\ 1) \qquad \text{for } i = 1, \ldots, n.$$

Moreover, the sum of n independent Chi-square random variables with one degree of freedom has a Chi-square distribution with n degrees of freedom, i.e.,

$$\chi^2\left(\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2;\ n\right).$$

Since the n-dimensional radial error random variable is defined by

$$R_n^2 = \sum_{i=1}^{n} X_i^2,$$

we see that

$$\chi^2(R_n^2/\sigma^2;\ n).$$

So the density function for R_n^2/σ^2 is

$$k_n(r^2/\sigma^2) = \frac{1}{\Gamma(n/2)2^{n/2}}\left(\frac{r^2}{\sigma^2}\right)^{n/2-1} e^{-r^2/2\sigma^2}.$$

By transformation of variable the density function for R_n is

$$f_n(r) = \left|\frac{d}{dr}\frac{r^2}{\sigma^2}\right| k_n(r^2/\sigma^2) = \frac{2r}{\sigma^2}\frac{1}{\Gamma(n/2)2^{n/2}}\left(\frac{r^2}{\sigma^2}\right)^{n/2-1} e^{-r^2/2\sigma^2}$$

$$= \frac{1}{\Gamma(n/2)2^{n/2-1}}\frac{r^{n-1}}{\sigma^n} e^{-r^2/2\sigma^2}.$$

Therefore the density function of the random variable R_n is

$$f_n(r) = c_n \frac{r^{n-1}}{\sigma^n} e^{-r^2/2\sigma^2} \qquad r \geq 0 \qquad (2)$$

where

$$c_n = \frac{1}{\Gamma(n/2)2^{n/2-1}}. \qquad (3)$$

In particular we have for n = 1, 2, 3

$$f_1(r) = \frac{1}{\Gamma(1/2)2^{-1/2}}\frac{1}{\sigma} e^{-r^2/2\sigma^2} = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma} e^{-r^2/2\sigma^2}$$

$$f_2(r) = \frac{1}{\Gamma(1)}\frac{r}{\sigma^2} e^{-r^2/2\sigma^2} = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$$

$$f_3(r) = \frac{1}{\Gamma(3/2)2^{1/2}}\frac{r^2}{\sigma^3} e^{-r^2/2\sigma^2} = \sqrt{\frac{2}{\pi}}\frac{r^2}{\sigma^3} e^{-r^2/2\sigma^2}$$

Since $\Gamma(1/2) = \pi^{1/2}$ and $\Gamma(t + 1) = t\Gamma(t)$ for $t > 0$.

The cumulative distribution of the random variable R_n is

$$F_n(r) = P(R_n \leq r) = \int_0^r f_n(u)du = \int_0^r c_n \frac{u^{n-1}}{\sigma^n} e^{-u^2/2\sigma^2} du.$$

Letting $t = u^2/2\sigma^2$ we get

$$F_n(r) = 2^{n/2} c_n \int_0^{r^2/2\sigma^2} t^{(n-2)/2} e^{-t} dt = \frac{2}{\Gamma(n/2)} \int_0^{r^2/2\sigma^2} t^{n/2-1} e^{-t} dt$$

$$= 2 \frac{\Gamma_{r^2/2\sigma^2}(n/2)}{\Gamma(n/2)}$$

where $\Gamma_t(u)$ is the incomplete Gamma function.

Therefore

$$F_n(r) = 2I(r^2/2\sigma^2, n/2) \qquad r \geq 0 \qquad (4)$$

where $I(t,u)$ is the incomplete Gamma function ratio.

In particular we have for $n = 1, 2, 3$

$$F_1(r) = 2I(r^2/2\sigma^2, 1/2)$$

$$F_2(r) = 2I(r^2/2\sigma^2, 1)$$

$$F_3(r) = 2I(r^2/2\sigma^2, 3/2).$$

However, we will derive other expressions for the cumulative distribution functions and density functions which will permit their graphs to be plotted more easily from available tables.

When $n = 1$ we have

$$F_1(r) = \int_0^r f_1(u)\, du = \int_0^r \sqrt{\frac{2}{\pi}}\, \frac{1}{\sigma}\, e^{-u^2/2\sigma^2}\, du.$$

Letting $t = u/\sigma$ we get

$$F_1(r) = \sqrt{\frac{2}{\pi}} \int_0^{r/\sigma} e^{-t^2/2}\, dt = \sqrt{2\pi}\sqrt{\frac{2}{\pi}} \int_0^{r/\sigma} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt.$$

Therefore the cumulative distribution function of R_1 is

$$F_1(r) = 2J(r/\sigma) \tag{5}$$

where

$$J(x) = \int_0^x g(t) dt.$$

Thus the density function of R_1 is

$$f_1(r) = \frac{2}{\sigma}\, g(r/\sigma). \tag{6}$$

When $n = 2$ we have

$$F_2(r) = \int_0^r f_2(u) du = \int_0^r \frac{u}{\sigma^2}\, e^{-u^2/2\sigma^2}\, du.$$

Letting $t = u^2/2\sigma^2$ we get

$$F_2(r) = \int_0^{r^2/2\sigma^2} e^{-t}\, dt = -e^{-t}\Big|_0^{r^2/2\sigma^2} = 1 - e^{-r^2/2\sigma^2}. \tag{7}$$

Therefore the cumulative distribution function of R_2 is

$$F_2(r) = 1 - \frac{\sigma^2}{r}\, f_2(r). \tag{8}$$

This can be rewritten as

$$F_2(r) = 1 - \sqrt{2\pi}\, g(r/\sigma). \tag{9}$$

Thus the density function of R_2 is

$$f_2(r) = -\frac{\sqrt{2\pi}}{\sigma}\, g^{(1)}(r/\sigma). \tag{10}$$

When $n = 3$ we have

$$F_3(r) = \int_0^r f_3(u)du = \int_0^r \sqrt{\frac{2}{\pi}}\, \frac{u^2}{\sigma^3}\, e^{-u^2/2\sigma^2} du.$$

Letting $t = u/\sigma$ we get

$$F_3(r) = \sqrt{\frac{2}{\pi}} \int_0^{r/\sigma} t^2\, e^{-t^2/2}\, dt.$$

Integration by parts gives

$$F_3(r) = \sqrt{\frac{2}{\pi}} \left\{ -\, te^{-t^2/2} \Big|_0^{r/\sigma} - \int_0^{r/\sigma} e^{-t^2/2}\, d(-t) \right\}$$

$$= \frac{2}{\pi} \left\{ -\, \frac{r}{\sigma}\, e^{-r^2/2\sigma^2} + \sqrt{2\pi} \int_0^{r/\sigma} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt \right\}$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{2\pi} \int_0^{r/\sigma} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt - \sqrt{\frac{2}{\pi}}\, \frac{r}{\sigma}\, e^{-r^2/2\sigma^2}$$

Therefore the cumulative distribution function of R_3 is

$$F_3(r) = 2J(r/\sigma) - \frac{\sigma^2}{r} f_3(r). \tag{11}$$

This can be rewritten as

$$F_3(r) = 2J(r/\sigma) + \sqrt{2\pi} \sqrt{\frac{2}{\pi}} \left(-\, \frac{r}{\sigma} \right) \frac{e^{-r^2/2\sigma^2}}{\sqrt{2\pi}}$$

$$= 2\left[J(r/\sigma) - Q_1(r/\sigma)\, g(r/\sigma)\right]$$

where $Q_k(t)$ is the Hermite polynomial of order k defined by

$$(-1)^k\, Q_k(t)\, g(t) = g^{(k)}(t) = \frac{d^k}{dt^k}\, g(t)$$

so that

$$g^{(1)}(t) = -\, Q_1(t)\, g(t).$$

Therefore the cumulative distribution function of R_3 is

$$F_3(r) = 2\left[J(r/\sigma) + g^{(1)}(r/\sigma)\right]. \qquad (12)$$

Thus the density function of R_3 is

$$f_3(r) = \frac{2}{\sigma}\left[g(r/\sigma) + g^{(2)}(r/\sigma)\right]. \qquad (13)$$

III. EXPECTATION AND STANDARD DEVIATION OF THE RADIAL ERROR

Next we find the expectation μ_n and the standard deviation σ_n of the random variable R_n. The first moment of R_n is

$$E(R_n) = \int_0^\infty r\, f_n(r)\, dr = \int_0^\infty r\, c_n \frac{r^{n-1}}{\sigma^n} e^{-r^2/2\sigma^2} dr = c_n \int_0^\infty \frac{r^n}{\sigma^n} e^{-r^2/2\sigma^2}\, dr.$$

Letting $t = r/(2\sigma)^{1/2}$ we get

$$E(R_n) = \sigma 2^{n/2 + 1/2} c_n \int_0^\infty t^n e^{-t^2}\, dt = \frac{\sigma 2^{n/2 + 1/2}}{\Gamma(n/2) 2^{n/2-1}} \frac{1}{2} \Gamma\left[(n + 1)/2\right]$$

$$= 2^{1/2} \frac{\Gamma\left[(n + 1)/2\right]}{\Gamma(n/2)} \sigma.$$

Therefore the expectation of R_n is

$$\mu_n = e_n \sigma \tag{14}$$

where

$$e_n = 2^{1/2} \frac{\Gamma\left[(n + 1)/2\right]}{\Gamma(n/2)}. \tag{15}$$

In particular we have for $n = 1, 2, 3$

$$\mu_1 = 2^{1/2} \frac{\Gamma(1)}{\Gamma(1/2)} \sigma = \sqrt{\frac{2}{\pi}}\, \sigma \doteq 0.7979\, \sigma$$

$$\mu_2 = 2^{1/2} \frac{\Gamma(3/2)}{\Gamma(1)} \sigma = \sqrt{\frac{\pi}{2}}\, \sigma \doteq 1.2533\, \sigma$$

$$\mu_3 = 2^{1/2} \frac{\Gamma(2)}{\Gamma(3/2)} \sigma = 2\sqrt{\frac{2}{\pi}}\, \sigma \doteq 1.5958\, \sigma.$$

The second moment of R_n is

$$E(R_n^2) = \int_0^\infty r^2 f_n(r)\,dr = \int_0^\infty r^2 c_n \frac{r^{n-1}}{\sigma^n} e^{-r^2/2\sigma^2}\,dr$$

$$= c_n \int_0^\infty \frac{r^{n+1}}{\sigma^n} e^{-r^2/2\sigma^2}\,dr.$$

Letting $t = r^2/2\sigma^2$ we get

$$E(R_n^2) = \sigma^2 2^{n/2} c_n \int_0^\infty t^{n/2} e^{-t}\,dt = \sigma^2\, 2^{n/2} c_n \Gamma(n/2 + 1)$$

$$= \frac{\sigma^2 2^{n/2}}{\Gamma(n/2)2^{n/2-1}} \Gamma(n/2 + 1) = \frac{1}{\Gamma(n/2)2^{-1}} \frac{n}{2}\Gamma(n/2)\,\sigma^2 = n\sigma^2.$$

Therefore the second moment is

$$E(R_n^2) = n\sigma^2. \tag{16}$$

The variance σ_n^2 of R_n is

$$V(R_n) = E(R_n^2) - E^2(R_n) = n\sigma^2 - (e_n\sigma)^2 = (n - e_n{}^2)\,\sigma^2.$$

Therefore

$$\sigma_n^2 = v_n\,\sigma^2 \tag{17}$$

where

$$v_n = n - 2\left\{\frac{\Gamma\left[(n + 1)/2\right]}{\Gamma(n/2)}\right\}^2. \tag{18}$$

In particular

$$\sigma_1^2 = \left\{1 - 2\left[\frac{\Gamma(1)}{\Gamma(1/2)}\right]^2\right\}\sigma^2 = (1 - \tfrac{2}{\pi})\,\sigma^2 \doteq 0.3634\,\sigma^2$$

$$\sigma_2^2 = \left\{2 - 2\left[\frac{\Gamma(3/2)}{\Gamma(1)}\right]^2\right\}\sigma^2 = (2 - \frac{\pi}{2})\ \sigma^2 \doteq 0.4292\ \sigma^2$$

$$\sigma_3^2 = \left\{3 - 2\left[\frac{\Gamma(2)}{\Gamma(3/2)}\right]^2\right\}\sigma^2 = (3 - \frac{8}{\pi})\ \sigma^2 \doteq 0.4535\ \sigma^2.$$

Therefore the standard deviation of R_n is

$$\sigma_n = d_n\ \sigma \tag{19}$$

where

$$d_n = v_n^{1/2}. \tag{20}$$

In particular we have for $n = 1, 2, 3$

$$\sigma_1 = (1 - \frac{2}{\pi})^{1/2}\ \sigma \doteq 0.6028\ \sigma$$

$$\sigma_2 = (2 - \frac{\pi}{2})^{1/2}\ \sigma \doteq 0.6551\ \sigma$$

$$\sigma_3 = (3 - \frac{8}{\pi})^{1/2}\ \sigma \doteq 0.6734\ \sigma.$$

IV. MEDIAN OF THE RADIAL ERROR

The median α_n of the random variable R_n is defined implicitly by the equation

$$P(R_n \leq \alpha_n) = F_n(\alpha_n) = 1/2.$$

We will solve this equation in the one, two, and three-dimensional cases.

When $n = 1$ we have from (5) that

$$F_1(r) = 2J(r/\sigma).$$

The median α_1 satisfies

$$1/2 = F_1(\alpha_1) = 2J(\alpha_1/\sigma)$$

or

$$J(\alpha_1/\sigma) = 1/4. \tag{21}$$

Hence from tables of the standard Gaussian distribution function we get the approximation

$$\alpha_1/\sigma \doteq 0.6745.$$

Therefore the median of R_1 is

$$\alpha_1 = a_1 \sigma \tag{22}$$

where

$$a_1 \doteq 0.6745. \tag{23}$$

When $n = 2$ we have from (9) that

$$F_2(r) = 1 - \sqrt{2\pi}\, g(r/\sigma).$$

The median α_2 satisfies

$$1/2 = F_2(\alpha_2) = 1 - \sqrt{2\pi}\, g(\alpha_2/\sigma).$$

Thus

$$g(\alpha_2/\sigma) = \frac{1}{2\sqrt{2\pi}} \doteq 0.19947.$$

Hence from tables of the standard Gaussian density function we get the approximation

$$\alpha_2/\sigma \doteq 1.774 .$$

Therefore the median of R_2 is

$$\alpha_2 = a_2\sigma \tag{24}$$

where

$$a_2 \doteq 1.1774 \tag{25}$$

We could have used (7) directly to get the exact expression

$$\alpha_2 = \sqrt{2 \log_e 2}\,\sigma. \tag{26}$$

From this we see that in terms of α_2 the cumulative distribution function of R_2 is

$$F_2(r) = P(R_2 \leq r) = 1 - 2^{-r^2/\alpha_2^2} . \tag{27}$$

It is interesting to note that no such exact expressions can be found when $n = 1, 3$ since termwise integration of the infinite series involved yields an infinite series whose root can not be given in closed form.

When $n = 3$ we have from (12) that

$$F_3(r) = 2\left[J(r/\sigma) + g^{(1)}(r/\sigma)\right] .$$

The median α_3 satisfies

$$1/2 = F_3(\alpha_3) = 2\left[J(\alpha_3/\sigma) + g^{(1)}(\alpha_3/\sigma)\right]$$

or

$$J(\alpha_3/\sigma) + g^{(1)}(\alpha_3/\sigma) = 1/4. \tag{28}$$

Hence from tables of the standard Gaussian density function we get the

approximation

$$\alpha_3/\sigma \doteq 1.5382.$$

Therefore the median of R_3 is

$$\alpha_3 = a_3\sigma \tag{29}$$

where

$$a_3 \doteq 1.5382. \tag{30}$$

V. STANDARD RADIAL ERROR

It is often convenient to use the ratio of the radial error to the Gaussian standard deviation rather than the radial error itself. Let the random variable

$$S_n = R_n/\sigma$$

denote the standard radial error. The cumulative distribution function of the random variable S_n is

$$H_n(s) = P(S_n \leq s) = P(R_n/\sigma \leq s) = P(R_n \leq \sigma s) = F_n(\sigma s)$$

Therefore, from (4) we get

$$H_n(s) = 2I(s^2/2,\ n/2). \qquad s \geq 0 \qquad (31)$$

Thus, by using (5), (9), and (12) we find that

$$H_1(s) = 2J(s)$$

$$H_2(s) = 1 - \sqrt{2\pi}\ g(s)$$

$$H_3(s) = 2\left[J(s) + g^{(1)}(s)\right].$$

These cumulative distribution functions are graphed in Fig. 1. The density function of the random variable S_n is

$$h_n(s) = \left|\frac{d}{ds} r\right| f_n(r) = \left|\frac{d}{ds} \sigma s\right| f_n(\sigma s) = \sigma f_n(\sigma s).$$

Therefore, from (2) we get

$$h_n(s) = c_n s^{n-1} e^{-s^2/2}. \qquad s \geq 0 \qquad (32)$$

Thus, by using (6), (10), and (13) we find that

$$h_1(s) = 2g(s)$$

$$h_2(s) = -\sqrt{2\pi}\ g^{(1)}(s)$$

$$h_3(s) = 2\left[g(s) + g^{(2)}(s)\right].$$

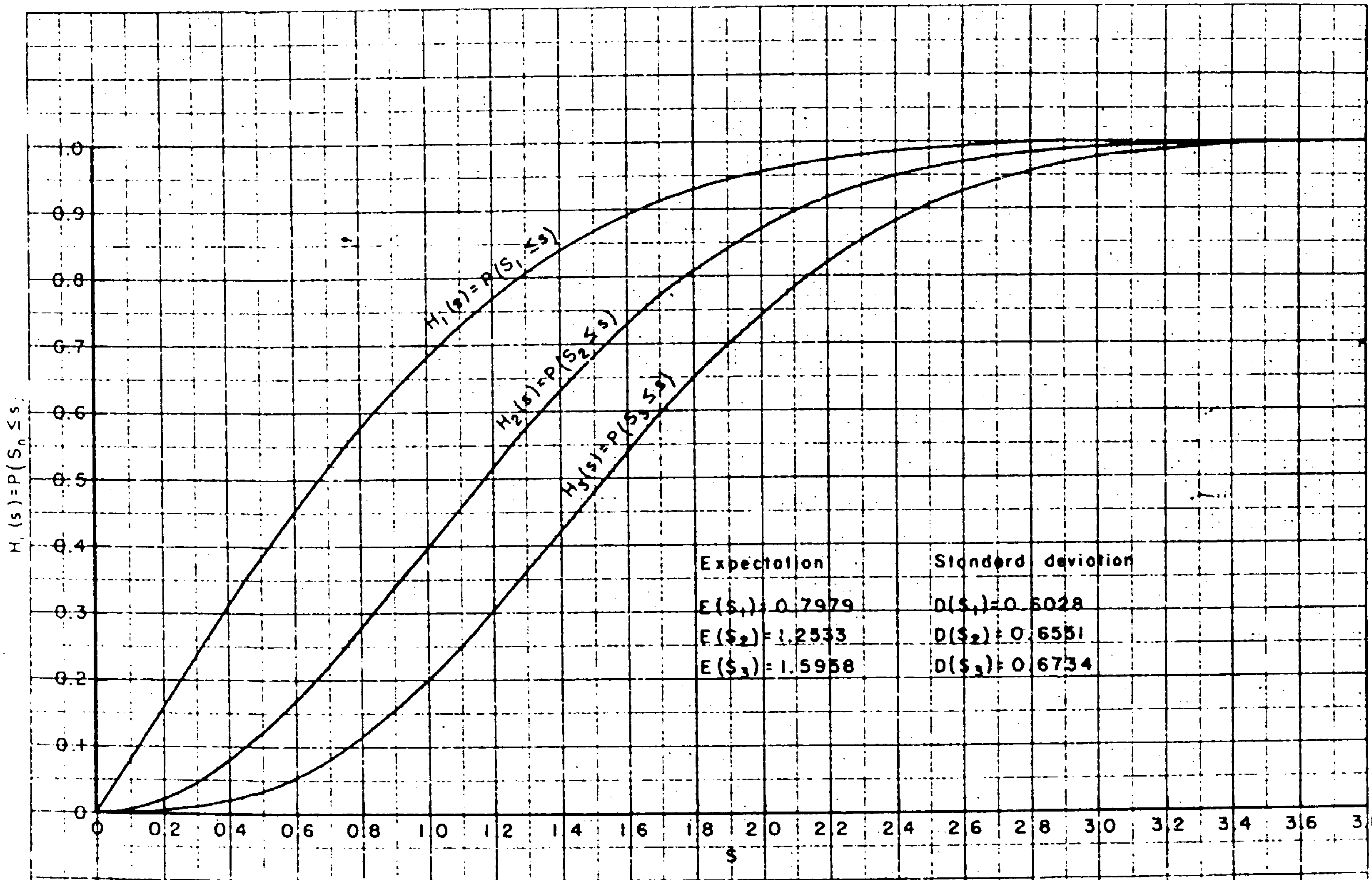

Fig. 1—Cumulative distribution function of standard radial error

These density functions are graphed in Fig. 2.

From (14) the expectation of S_n is

$$E(S_n) = e_n \tag{33}$$

From (17) and (19) the variance and standard deviation of S_n are, respectively,

$$V(S_n) = v_n \tag{34}$$

and

$$D(S_n) = d_n. \tag{35}$$

From (22), (24), and (27) the median of S_n is

$$M(S_n) = a_n. \tag{36}$$

Table 1 summarizes the pertinent statistical constants and Table 2 gives the conversion factors that inter-relate these constants.

Table 1

Statistical Constants

n	e_n	v_n	d_n	a_n
1	0.7979	0.3634	0.6028	0.6745
2	1.2533	0.4292	0.6551	1.1774
3	1.5958	0.4535	0.6734	1.5382

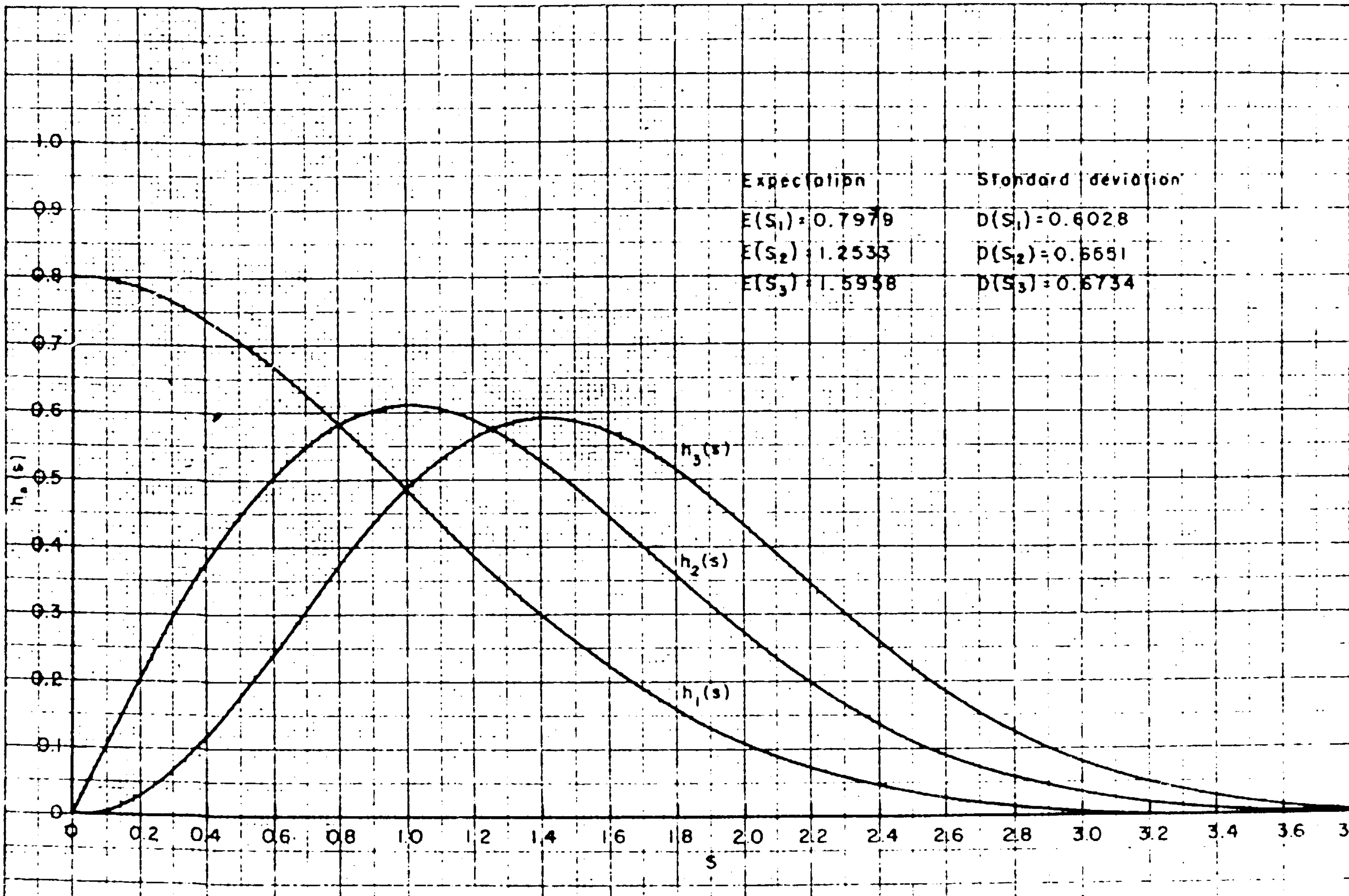

Fig. 2 — Density function of standard radial error

Table 2

Conversion Factors

$$\mu_1 = 0.7979\sigma = 1.1830\alpha_1$$

$$\alpha_1 = 0.6745\sigma = 0.8453\mu_1$$

$$\sigma = 1.4826\alpha_1 = 1.2533\mu_1$$

$$\mu_2 = 1.2533\sigma = 1.0645\alpha_2$$

$$\alpha_2 = 1.1774\sigma = 0.9394\mu_2 = 1.7456\alpha_1$$

$$\sigma = 0.8493\alpha_2 = 0.7979\mu_2$$

$$\mu_3 = 1.5958\sigma = 1.0374\alpha_3$$

$$\alpha_3 = 1.5382\sigma = 0.9639\mu_3 = 2.2805\alpha_1$$

$$\sigma = 0.6501\alpha_3 = 0.6267\mu_3$$

VI. STATISTICAL APPLICATION IN WAR GAMING

For war gaming purposes, the detonation point of the weapon can be determined by random sampling from just three inputs:

(1) aiming point

(2) direction of approach

(3) means of delivery.

These give, respectively:

(1') origin of the coordinate system

(2') orientation of the coordinate system

(3') median radial error of the weapon.

From the aiming point the origin of an n-dimensional rectangular coordinate system $Y_1, \ldots, Y_n$ is located. From the direction of approach to the target the axes $Y_1, \ldots, Y_n$ are oriented so that one of them, say Y_1, lies along the approach direction. From the means of delivery of the weapon the median radial error is chosen. The median radial error is the radius of the n-sphere within which the weapon has a $50^{\circ}/o$ probability of detonating. Now it is assumed that the radial error has an n-dimensional spherical Gaussian distribution centered at the aiming point as an origin and scaled by the standard deviation σ. But it was shown earler that α_n is related to σ by the equation

$$\alpha_n = a_n \sigma \tag{37}$$

where a_n is a known constant and σ is the standard deviation of the one-dimensional Gaussian distributions that compose the n-dimensional spherical Gaussian distribution. Hence, knowing α_n and a_n we solve (37) for σ. This determines which Gaussian curve governs the distribution of the detonation point along each of the axes $Y_1, \ldots, Y_n$.

From a table of standard (i.e., zero expectation and unit variance) Gaussian random numbers we select n numbers $x_1, \ldots, x_n$. Now if $\sigma = 1$ these numbers would suffice, but in general we must use σ as a scaling constant and determine the rectangular coordinates $y_1, \ldots, y_n$ of the detonation point from the equations

$$y_i = \sigma x_i \qquad i = 1, \ldots, n \qquad (38)$$

Because of (22), (38) becomes

$$y_i = \frac{\alpha_n}{a_n} x_i \qquad i = 1, \ldots, n \qquad (39)$$

and we plot the desired detonation point $(y_1, \ldots, y_n)$.

The following example illustrates the method when $n = 2$. Given an aiming point, a direction of approach from West to East, and a CEP of 1500 feet, we statistically determine the detonation point as follows:

From (37) we find

$$\sigma = \frac{\alpha_2}{a_2} = \frac{1500 \text{ ft}}{1.18} = 1270 \text{ ft.}$$

Selecting two numbers from the table of standard Gaussian random numbers we get

$$x_1 = -1.030$$
$$x_2 = 0.537$$

Hence from (39) we have

$$y_1 = \sigma x_1 = -1310 \text{ ft}$$
$$y_2 = \sigma x_2 = 683 \text{ ft}$$

which are plotted in Fig. 3.

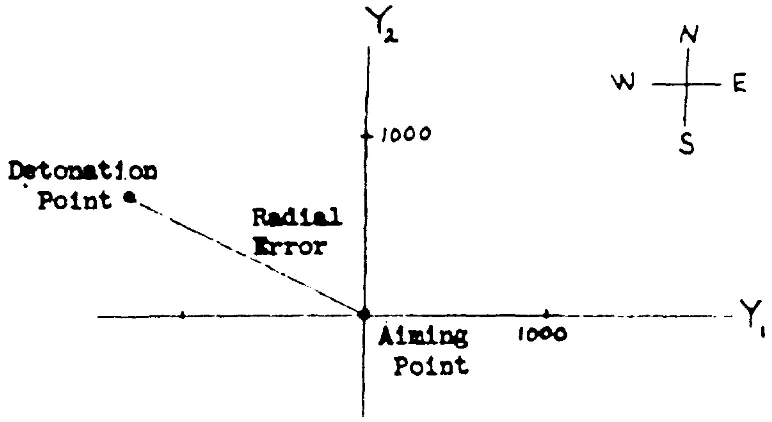

Fig. 3 - Plot of radial error when n = 2

The method for three-dimensional weapons such as rockets or torpedoes is analogous. When n = 3 we find from (37) that

$$\sigma = \frac{\alpha_3}{a_3} = \frac{\alpha_3}{1.54}$$

is the proper relation. We select three numbers x_1, x_2, x_3 from the table of standard Gaussian random numbers. They are converted to y_1, y_2, y_3 by means of (39). Finally, the detonation point (y_1, y_2, y_3) is plotted in the three-dimensional space with origin at the aiming point and with the Y_1 axis along the approach direction.

The foregoing statistical method applies to non-gross weapon errors. We might, for war gaming purposes, define a gross error by means of the equations

$$y_i = 5\sigma x_i \qquad i = 1, \ldots, n \qquad (40)$$

and regard it as a random event following some given probability distribution. For example, if it is assumed that the probability of a gross error is 0.1, then a table of uniform (i.e., equi-probable) random digits is used to sample statistically as follows. If m weapons are to be used, then m digits are selected independently, and only if a digit is zero (say) is the corresponding attack ruled a gross error.

REFERENCES

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